

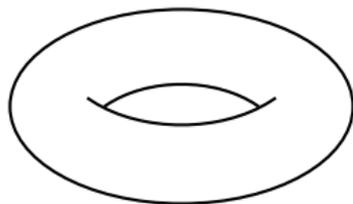
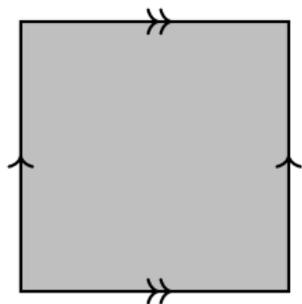
Equivariant surfaces and $RO(G)$ -graded cohomology

Clover May

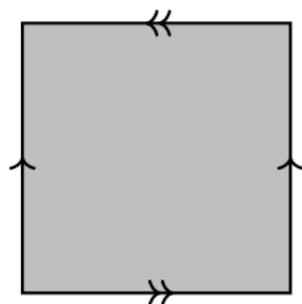
27 January 2026



Some surfaces



Torus



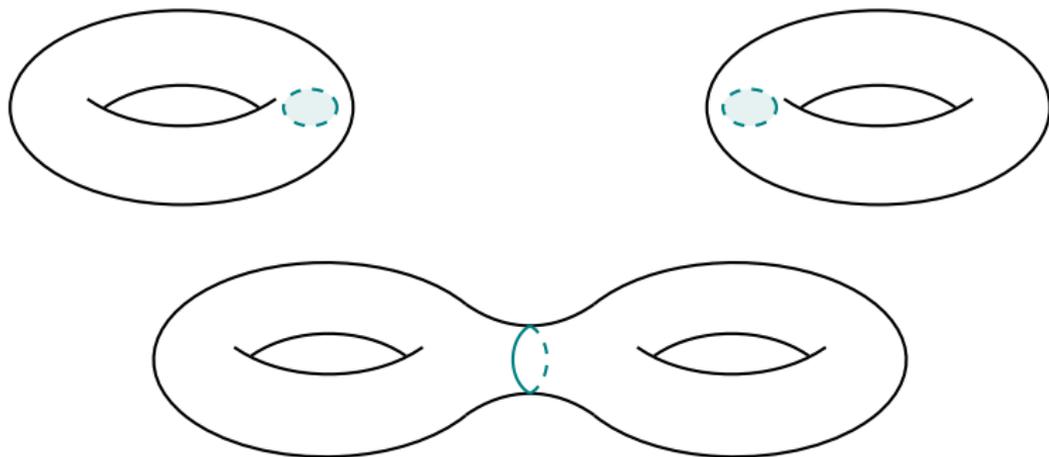
Klein bottle (non-orientable)

Build new surfaces by gluing

Given two surfaces S_1 and S_2

- cut a small disk out of each surface
- glue the edges of the holes together

This forms a new surface $S_1 \# S_2$ called the **connected sum**



$$T_2 = T \# T$$

Classification of surfaces

Classical result due to work of a number of people

- Early versions due to Möbius (1861) and Jordan (1866)
- More detailed proofs by von Dyck (1888) and Dehn–Heegard (1907)
- Rigorous proof by Brahma (1921)

Theorem

Up to homeomorphism, every compact surface (closed and bounded with no boundary) is

- *a sphere S^2 ,*
- *a connected sum of tori $T_g = T \# T \# \cdots \# T$, or*
- *a connected sum of real projective spaces $N_r = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$.*

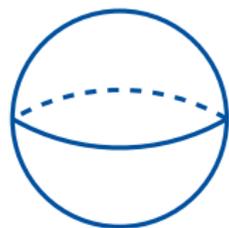
Equivariant homotopy

G - finite group

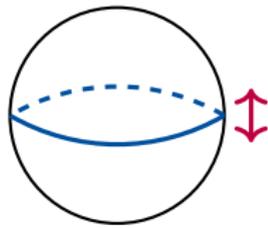
- Capture symmetries of spaces by considering G -actions on spaces up to equivariant homeomorphism

$$G = C_2$$

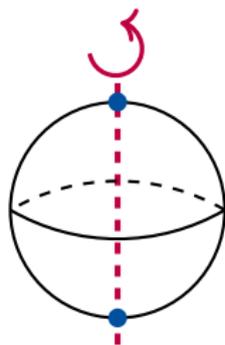
- C_2 -action also called an **involution**
- Involutions on a sphere



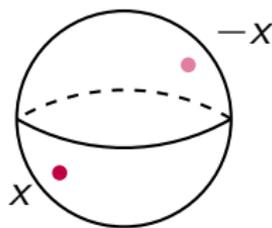
S^2_{triv}



S^2_{flip}



S^2_{rot}

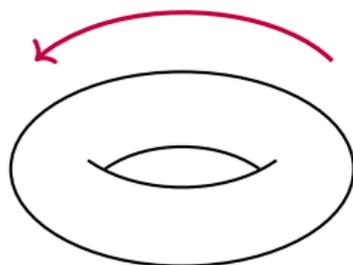


S^2_{anti}

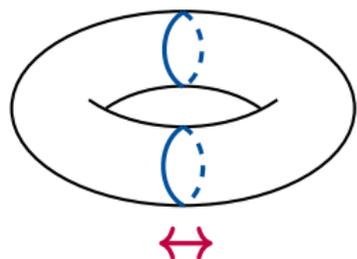
Involutions on a torus



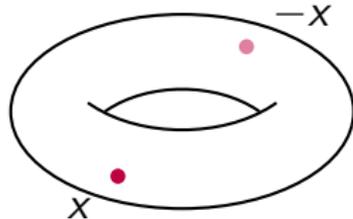
T_{triv}



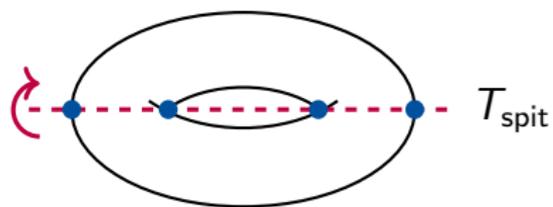
T_{rot}



T_{flip}



T_{anti}

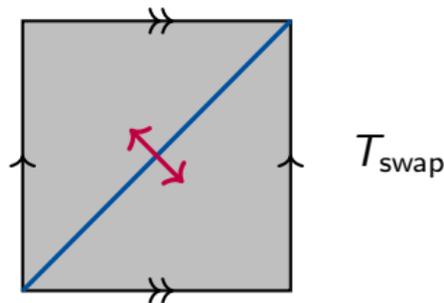


T_{spit}

Classification of C_2 -actions on surfaces

Up to isomorphism, there are exactly six involutions on a torus.

The last one



Theorem (Dugger 2019)

Up to isomorphism, there are exactly $4 + 2g$ involutions on the genus g torus T_g .

Moreover, gave a complete classification of isomorphism classes of involutions on compact surfaces using equivariant surgery.

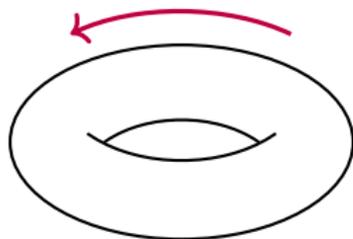
C_3 -actions on a torus

Theorem (Pohland 2023)

Up to isomorphism, there are three C_3 -actions on a torus.

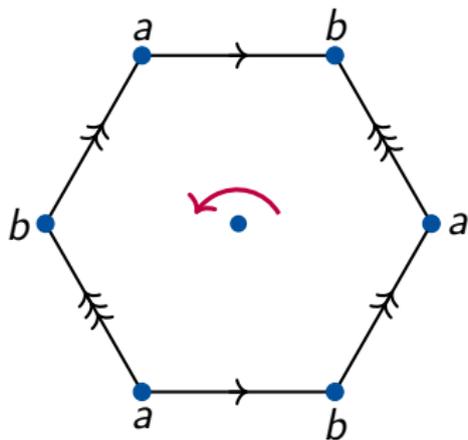


T_{triv}



T_{rot}

The last one



Classification of C_p -actions on surfaces

Theorem (Pohland 2023)

For p an odd prime, gave a complete classification of isomorphism classes of C_p -actions on compact surfaces using equivariant surgery.

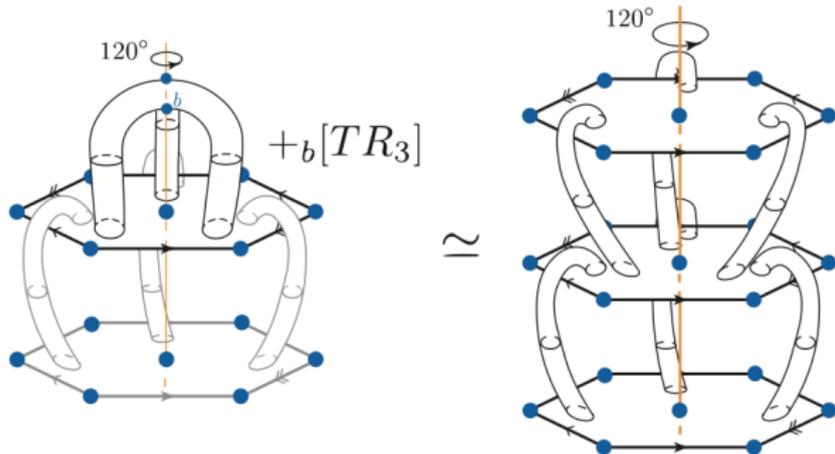


FIGURE 15. Twisted ribbon surgery centered on b yields the space Poly_3 .

Classical cohomology

Nonequivariant

- CW-complex X
- Attach cells D^n along boundary $\partial D^n = S^{n-1}$
- Cellular cohomology $H^*(X)$
- Suspension $\Sigma^m X = S^m \wedge X = (S^m \times X)/(S^m \vee X)$
- Suspension isomorphism $\tilde{H}^n(X) \cong \tilde{H}^{n+m}(\Sigma^m X)$
- $H^*(pt; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & * = 0 \\ 0 & \text{else} \end{cases}$
- Represented by an Eilenberg–MacLane spectrum $H\mathbb{F}_2$

$RO(G)$ -graded cohomology

G - finite group

- G -CW complex
- Attach orbit cells $G/K \times D^n$, for $K \leq G$
- Integer graded Bredon cohomology $H_G^*(-)$
- Coefficient system $H_G^*(G/K) \longrightarrow H_G^*(G/J)$

- V - real representation of G
- $S^V = \widehat{V}$ one-point compactification
- Suspension $\Sigma^V X = S^V \wedge X$

$RO(G)$ -graded cohomology

- $RO(G) = \mathbb{Z}\langle \text{irreducible real orthogonal } G\text{-reps} \rangle$
- Coefficients in a Mackey functor

$$H_G^*(G/K) \xrightarrow{\quad} H_G^*(G/J)$$

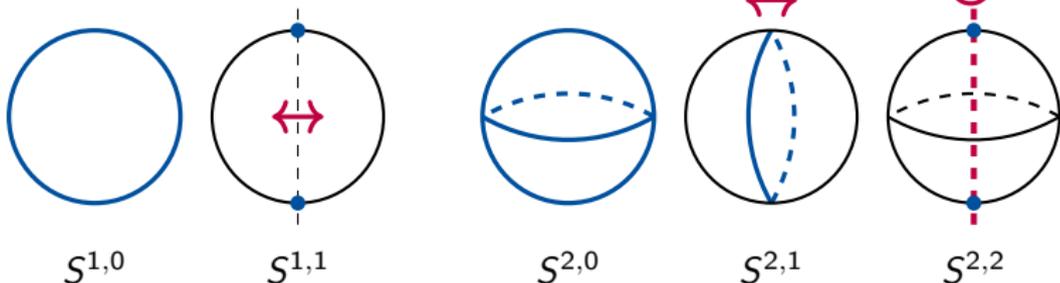

- For $\alpha \in RO(G)$ any virtual representation and M a Mackey functor, get $H_G^\alpha(-; M)$
- Suspension isomorphism $\tilde{H}_G^\alpha(X; M) \cong \tilde{H}_G^{\alpha+V}(\Sigma^V X; M)$
- Represented by an equivariant Eilenberg–MacLane spectrum HM
- Note - different from Borel equivariant cohomology

$$H_{Bor}^*(X) := H^*(EG \times_G X)$$

$RO(C_2)$ -graded cohomology

$$G = C_2$$

- Two orbits: $pt = C_2/C_2$ and $C_2 = C_2/e$
- Representations $V = \mathbb{R}^{p,q} = (\mathbb{R}_{triv})^{p-q} \oplus (\mathbb{R}_{sgn})^q$
- Representation spheres $S^V = S^{p,q}$



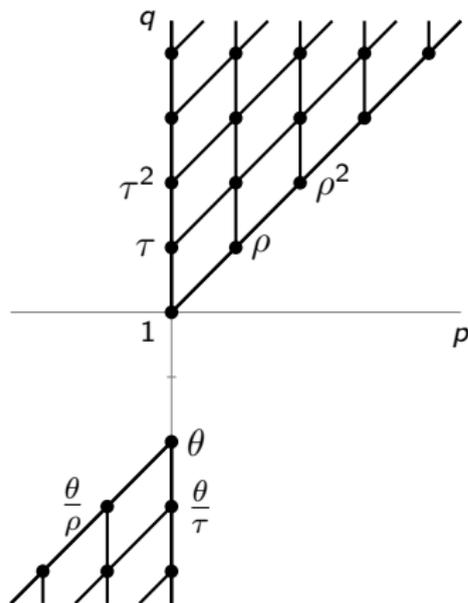
- Coefficients in the constant Mackey functor: $\underline{\mathbb{F}}_2$
- Write $H_G^\alpha(X; \underline{\mathbb{F}}_2) = H_{C_2}^{p,q}(X; \underline{\mathbb{F}}_2)$
- Represented by an Eilenberg–MacLane C_2 -spectrum $H\underline{\mathbb{F}}_2$

Cohomology of a point

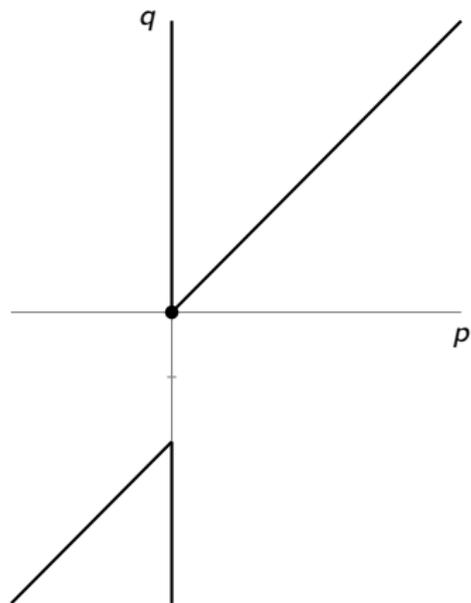
• = \mathbb{F}_2

| $\cdot \tau$

/ $\cdot \rho$



$$\mathbb{M}_2 = H_{C_2}^{*,*}(pt; \mathbb{F}_2)$$

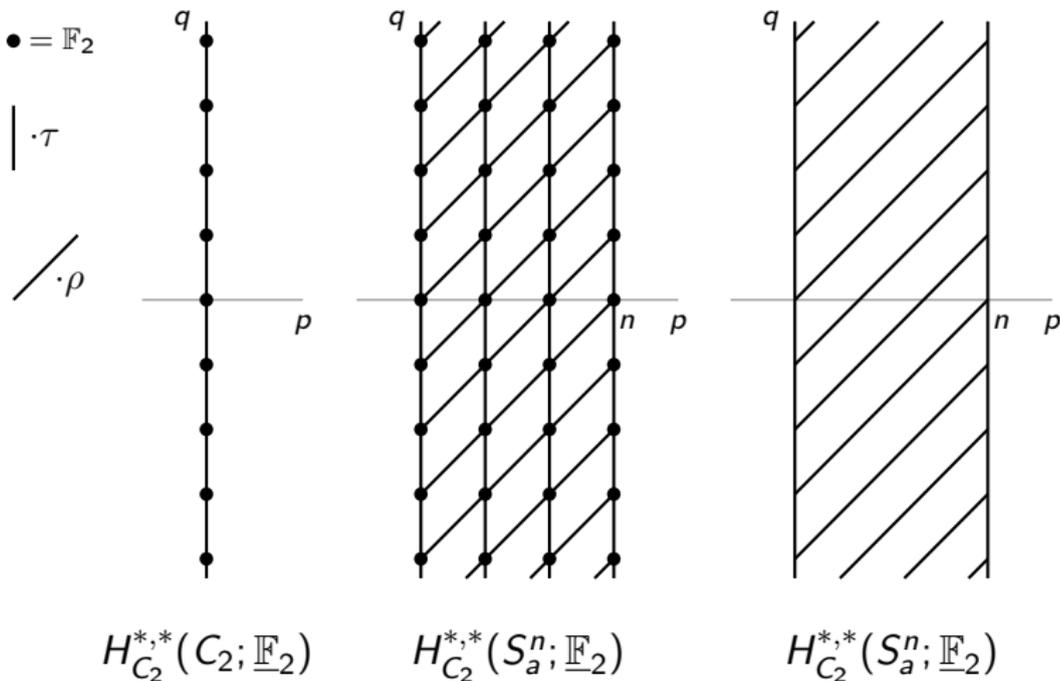


$$\mathbb{M}_2 = H_{C_2}^{*,*}(pt; \mathbb{F}_2)$$

$$\tilde{H}_{C_2}^{*,*}(S^{p,q}; \mathbb{F}_2) \cong \Sigma^{p,q} \mathbb{M}_2$$

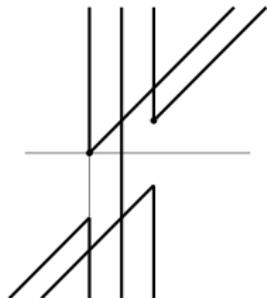
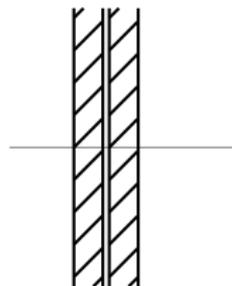
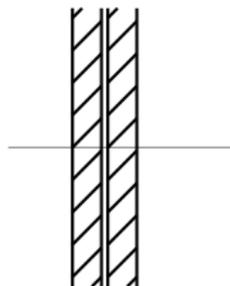
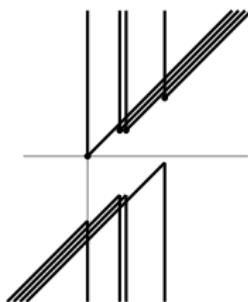
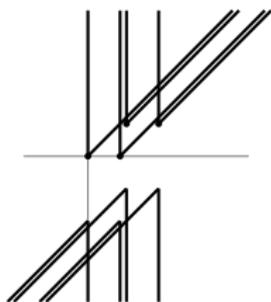
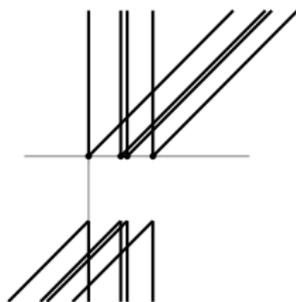
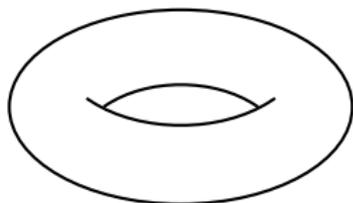
Examples

For any X , $H_{C_2}^{*,*}(X; \underline{\mathbb{F}}_2)$ is an \mathbb{M}_2 -module via $X \rightarrow pt$



Torus examples

Cohomologies of C_2 -actions on a torus with \mathbb{F}_2 -coefficients

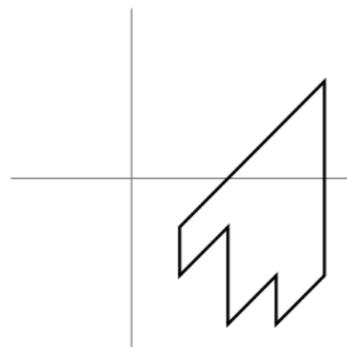


Structure theorem

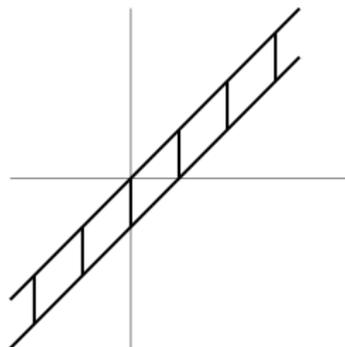
Theorem (M. 2020)

For X a finite C_2 -CW complex, $H_{C_2}^{*,*}(X; \mathbb{F}_2)$ splits as a direct sum of shifted copies of

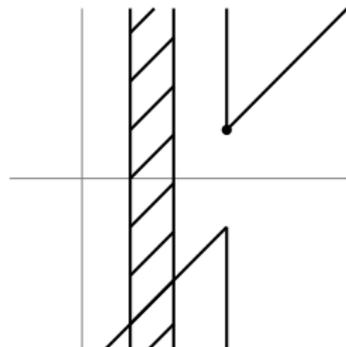
$$\mathbb{M}_2 = H_{C_2}^{*,*}(pt) \quad \text{and} \quad H_{C_2}^{*,*}(S_a^n).$$



X



X



✓

Structure theorem applications

Theorem (M. 2020)

For X a finite C_2 -CW complex, $H_{C_2}^{*,*}(X; \mathbb{F}_2)$ splits as a direct sum of shifted copies of

$$\mathbb{M}_2 = H_{C_2}^{*,*}(pt) \quad \text{and} \quad H_{C_2}^{*,*}(S_a^n).$$

Some applications

- [Hazel 2021, 2023] Complete computations of cohomology of C_2 -surfaces using Dugger's classification
- [Pitsch–Ricka–Scherer 2021] Characterization of conjugation spaces
- [dos Santos–Florentino–Orts 2025] Classification of maximal and Galois-maximal real algebraic varieties

Lifting to C_2 -spectra

Theorem (M. 2020)

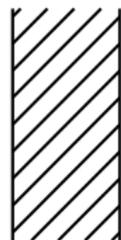
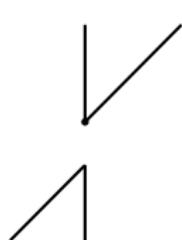
For Y a finite C_2 -CW spectrum, $Y \wedge H\underline{\mathbb{F}}_2$ splits as a wedge of (rep) suspensions of

$$H\underline{\mathbb{F}}_2 \quad \text{and} \quad (S_a^n)_+ \wedge H\underline{\mathbb{F}}_2.$$

Theorem (Dugger–Hazel–M. 2024)

Any compact C_2 -equivariant $H\underline{\mathbb{F}}_2$ -module splits as a wedge of (rep) suspensions of

$$H\underline{\mathbb{F}}_2, \quad (S_a^n)_+ \wedge H\underline{\mathbb{F}}_2, \quad \text{and} \quad \text{cof}(\tau^m).$$



At odd primes

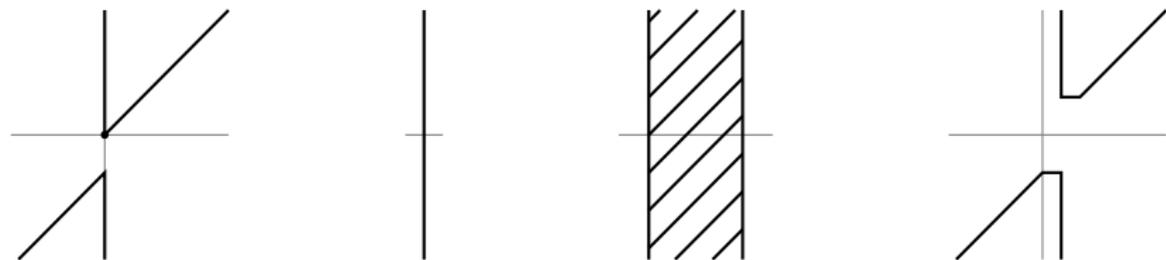
Theorem (Pohland 2025)

Complete computations of $RO(C_3)$ -graded cohomology of C_3 -surfaces with \mathbb{F}_3 -coefficients.

Conjecture (M. 2019)

For X a finite C_3 -CW complex, $H_{C_3}^{,*}(X; \mathbb{F}_3)$ splits as a direct sum of shifted copies of*

$$\mathbb{M}_3 = H_{C_3}^{*,*}(pt), \quad H_{C_3}^{*,*}(C_3), \quad H_{C_3}^{*,*}(S_{free}^{2n+1}), \quad \text{and} \quad \tilde{H}_{C_3}^{*,*}(EB).$$



At odd primes

Theorem (Grevstad–M. 2025)

*The classification of C_p -equivariant compact $H\mathbb{F}_p$ -modules is **wild** for p an odd prime.*

Morally

- impossible to describe all indecomposables
- such a list contains every indecomposable module of every finite-dimensional \mathbb{F}_p -algebra

Example

$\mathbb{F}_p[C_p \times C_p]$ is wild unless $p = 2$

Theorem (Grevstad–M. 2025)

*The classification of G -equivariant compact $H\mathbb{F}_p$ -modules is **wild** whenever G surjects onto a p -group of order more than 2.*

Thank you!